# On diffeomorphism extension

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**Abstract**

We state and prove a result concerning the fact that a diffeomorphism can be "extended" in such a way that its image is $\mathbb{R}^n$.

Before stating the theorem, we introduce the following property :

**Definition 1 (Conditions $(\mathcal{C})$)** *An open subset $E$ of $\mathbb{R}^m$ is said to verify condition $(\mathcal{C})$ if there exist a $C^1$ function $\kappa : \mathbb{R}^m \to \mathbb{R}$, a bounded[1] $C^1$ vector field $\chi$, and a closed set $K_0$ contained in $E$ such that:*

1. $E = \{z \in \mathbb{R}^n, \kappa(z) < 0\}$
2. $K_0$ *is globally attractive for* $\chi$
3. *we have the following transversality assumption:*

$$\frac{\partial \kappa}{\partial z}(z)\chi(z) < 0 \qquad \forall z \in \mathbb{R}^m : \ \kappa(z) = 0.$$

We now state the main theorem of this note :

**Theorem 1 (Image extension)** *Let $\psi$: $\mathcal{D} \subset \mathbb{R}^m \to \psi(\mathcal{D}) \subset \mathbb{R}^m$ be a diffeomorphism. If $\psi(\mathcal{D})$ verifies condition $(\mathcal{C})$ or $\mathcal{D}$ is $C^2$-diffeomorphic to $\mathbb{R}^m$ and $\psi$ is $C^2$, then for any compact set $K$ in $\mathcal{D}$ there exists a diffeomorphism $\psi_e : \mathcal{D} \to \mathbb{R}^m$ satisfying :*

$$\psi_e(\mathcal{D}) = \mathbb{R}^m \quad , \qquad \psi_e(z) = \psi(z) \quad \forall z \in K.$$

The proof of this theorem, given in Section 2, relies on two preliminary lemmas presented in Section 1.

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[1] If not replace $\chi$ by $\frac{\chi}{\sqrt{1+|\chi|^2}}$.

# 1 Technical lemmas

## 1.1 Construction of a diffeomorphism from an open set to $\mathbb{R}^m$

In this section, we give sufficient conditions to build a diffeomorphism from $\mathbb{R}^m$ to an open subset $E$ which leaves $E_\varepsilon \subset E$ unchanged. This construction is made explicit in the proof.

The complementary, closure, interior and boundary of a set $S$ are denoted $S^c$, $\overline{S}$, $\overset{\circ}{S}$ and $\partial S$, respectively, with $\partial S = \overline{S} \backslash \overset{\circ}{S}$.

The Hausdorff distance $d_H$ between two sets $A$ and $B$ is defined by :

$$d_H(A,B) \;=\; \max\left\{ \sup_{x_A \in A} \inf_{x_B \in B} |x_A - x_B| \,, \sup_{x \in A} \inf_{x_B \in B} |x_A - x_B| \right\}$$

With $Z(z,t)$ we denote the (unique) solution, at time $t$, to $\dot{z} = \chi(z)$ going trough $z$ at time 0.

**Lemma 1** *Let $E$ be an open strict subset of $\mathbb{R}^m$ verifying $(\mathcal{C})$, with a $C^s$ vector field $\chi$. Then, for any strictly positive real number $\varepsilon$, there exists a $C^s$-diffeomorphism $\phi$: $\mathbb{R}^m \to E$, such that, with*

$$\Sigma = \bigcup_{t\in[0,\varepsilon]} Z(\partial E, t) \ ,$$

*we have $\phi(z) = z$ for all $z \in E_\varepsilon = E \cap \Sigma^{\mathsf{c}}$ and $d_H(\partial E_\varepsilon, \partial E) \;\leq\; \varepsilon \sup_z |\chi(z)|$.*

**Proof :** We start by establishing some properties.

– *$E$ is forward invariant by $\chi$.* This is a direct consequence of points 1 and 3 of the condition $(\mathcal{C})$.

– *$\Sigma$ is closed.* Take a sequence $(z_k)$ of points in $\Sigma$ converging to $z^*$. By definition, there exists a sequence $(t_k)$, such that :

$$t_k \in [0,\varepsilon] \qquad \text{and} \qquad Z(z_k, -t_k) \in \partial E \qquad \forall k \in \mathbb{N} \ .$$

Since $[0,\varepsilon]$ is compact, one can extract a subsequence $(t_{\sigma(k)})$ converging to $t^*$ in $[0,\varepsilon]$, and by continuity of the function $(z,t) \mapsto Z(z,-t)$, $(Z(z_{\sigma(k)}, t_{\sigma(k)}))$ tends to $Z(z^*, -t^*)$ which is in $\partial E$, since $\partial E$ is closed. Finally, because $t^*$ is in $[0,\varepsilon]$, $z^*$ is in $\Sigma$ by definition.

– *$\Sigma$ is contained in $\mathtt{cl}(E)$.* Since, $E$ is forward invariant by $\chi$, and so is $\mathtt{cl}(E)$ (see [4, Theorem 16.3]). This implies

$$\partial E \;\subset\; \Sigma = \bigcup_{t\in[0,\varepsilon]} Z(\partial E, t) \;\subset\; \mathtt{cl}(E) \;=\; E \cup \partial E \ .$$

At this point, it is useful to note that, because $\Sigma$ is a closed subset of the open set $E$, we have $\Sigma \cap E \;=\; \Sigma \backslash \partial E$. This implies :

$$E \backslash E_\varepsilon = (E_\varepsilon)^{\mathsf{c}} \cap E = (E^{\mathsf{c}} \cup \Sigma) \cap E = \Sigma \cap E = \Sigma \backslash \partial E, \tag{1}$$

and $E = E_\varepsilon \underset{\neq}{\cup} (\Sigma \backslash \partial E)$.

With all these properties at hand, we define now two functions $t_z$ and $\theta_z$. The assumptions of global attractiveness of the closed set $K_0$ contained in $E$ open, of transversality of $\chi$ to $\partial E$, and the property of forward-invariance of $E$, imply that, for all $z$ in $E^{\mathsf{c}}$, there exists a unique non negative real number $t_z$ satisfying:

$$\kappa\left(Z(z,t_z)\right)=0 \quad \Longleftrightarrow \quad Z(z,t_z)\in\partial E.$$

The same arguments in reverse time allow us to see that, for all $z$ in $\Sigma$, $t_z$ exists, is unique and in $[-\varepsilon,0]$. This way, the function $z\to t_z$ is defined on $(E_\varepsilon)^{\mathsf{c}}$. Next, for all $z$ in $(E_\varepsilon)^{\mathsf{c}}$, we define :

$$\theta_z = Z(z,t_z).$$

Thanks to the transversality assumption, the Implicit Function Theorem implies the functions $z\mapsto t_z$ and $z\mapsto\theta_z$ are $C^s$ on $(E_\varepsilon)^{\mathsf{c}}$.

**Remark 1** $\kappa$ having constant rank 1 in a neighborhood of $\partial E$, this set is a closed, regular submanifold of $\mathbb{R}^m$. The arguments above show that $z\mapsto(\theta_z,t_z)$ is a diffeomorphism between $E^{\mathsf{c}}$ and $\partial E\times[0,+\infty[$. Since $\partial E$ is a deformation retract of $E^{\mathsf{c}}$ and the open unit ball is diffeomorphic to $\mathbb{R}^m$ [**?**], if $E$ were bounded, $E^{\mathsf{c}}$ could be seen as a $h$-cobordism between $\partial E$ and the unit sphere $\mathbb{S}^{m-1}$ and $t_z$ as a Morse function with no critical point in $E^{\mathsf{c}}$. See [5] for instance.

Now we evaluate $t_z$ for $z$ in $\partial\Sigma$. Let $z$ be arbitrary in $\partial\Sigma$ and therefore in $\Sigma$ which is closed. Assume its corresponding $t_z$ is in $]-\varepsilon,0[$. The Implicit Function Theorem shows that $z\mapsto t_z$ and $z\mapsto\theta_z$ are defined and continuous on a neighborhood of $z$. Therefore, there exists a strictly positive real number $r$ satisfying

$$\forall y\in\mathcal{B}_r(z)\ ,\ \exists t_y\in]-\varepsilon,0[:\ Z(y,t_y)\in\partial E\ .$$

This implies that the neighborhood $\mathcal{B}_r(z)$ of $z$ is contained in $\Sigma$, in contradiction with the fact that $z$ is on the boundary of $\Sigma$.

This shows that, for all $z$ in $\partial\Sigma$, $t_z$ is either 0 or $-\varepsilon$. We write this as $\partial\Sigma=\partial E\cup(\partial\Sigma)_i$, with the notation $(\partial\Sigma)_i=\left\{z\in\Sigma\ :\ t_z=-\varepsilon\right\}$.

Now we want to prove $\partial E_\varepsilon\subset(\partial\Sigma)_i$. To obtain this result, we start by showing :

$$\partial E_\varepsilon\cap\partial E=\emptyset \quad\text{and}\quad \partial E_\varepsilon\subset\partial\Sigma\ . \tag{2}$$

Suppose the existence of $z$ in $\partial E_\varepsilon\cap\partial E$. $z$ being in $\partial E$, its corresponding $t_z$ is 0. By the Implicit Function Theorem, there exists a strictly positive real number $r$ such that,

$$\forall y\in\mathcal{B}_r(z)\ ,\ \exists t_y\in\left]-\tfrac{\varepsilon}{2},\tfrac{\varepsilon}{2}\right[\ :Z(y,t_y)\in\partial E\ .$$

But, by definition, any $y$, for which there exists $t_y$ in $]\frac{-\varepsilon}{2},0]$, is in $\Sigma$. If instead $t_y$ is strictly positive, then necessarily $y$ is in $E^{\mathsf{c}}$, because $E$ is forward-invariant by $\chi$ and a solution starting in $E$ cannot reach $\partial E$ in positive finite time. We have obtained : $\mathcal{B}_r(z)\subset\Sigma\cup E^{\mathsf{c}}=(E_\varepsilon)^{\mathsf{c}}$. $\mathcal{B}_r(z)$ being a neighborhood of $z$, this contradicts the fact that $z$ is in the boundary of $E_\varepsilon$.

At this point, we have proved that $\partial E_\varepsilon\cap\partial E=\emptyset$, and, because $E_\varepsilon$ is contained in $E$, this implies $\partial E_\varepsilon\subset E$. With this, (2) will be established by proving that we have $\partial E_\varepsilon\subset\partial\Sigma$. Let $z$ be arbitrary in $\partial E_\varepsilon$ and therefore in $E$ which is open. There exists a strictly positive real number $r$ such that we have :

$$\mathcal{B}_r(z)\cap E_\varepsilon\ =\ \mathcal{B}_r(z)\ \cap\ (E\cap\Sigma^{\mathsf{c}})\neq\emptyset\ ,\ \mathcal{B}_r(z)\cap E_\varepsilon^{\mathsf{c}}\ =\ \mathcal{B}_r(z)\ \cap\ (E^{\mathsf{c}}\cup\Sigma)\neq\emptyset\ ,\ \mathcal{B}_r(z)\subset E\ .$$

This implies $\mathcal{B}_r(z) \cap \Sigma^c \neq \emptyset$ and $\mathcal{B}_r(z) \cap \Sigma \neq \emptyset$ and therefore that $z$ is in $\partial\Sigma$.

We have established $\partial E_\varepsilon \cap \partial E = \emptyset$, $\partial E_\varepsilon \subset \partial\Sigma$ and $\partial\Sigma = \partial E \cup (\partial\Sigma)_i$. This does imply :

$$\partial E_\varepsilon \subset (\partial\Sigma)_i = \{z \in E : t_z = -\varepsilon\} . \tag{3}$$

This allows us to extend by continuity the definition of $t_z$ to $\mathbb{R}^m$ by letting $t_z = -\varepsilon$ for all $z \in E_\varepsilon$.

Thanks to all these preparatory steps, we are finally ready to define a function $\phi : \mathbb{R}^m \to E$ as:

$$\phi(z) = \begin{cases} Z\left(z, t_z + \nu(t_z)\right), & \text{if } z \in (E_\varepsilon)^c , \\ z, & \text{if } z \in E_\varepsilon , \end{cases} \tag{4}$$

where $\nu$ is an arbitrary $C^s$ and strictly decreasing function defined on $\mathbb{R}$ satisfying:

$$\nu(t) = -t \quad \forall t \leq -\varepsilon \quad , \qquad \lim_{t \to +\infty} \nu(t) = 0.$$

The image of $\phi$ is contained in $E$ since we have $E_\varepsilon \subset E$ and :

$$\begin{gathered} t_z + \nu(t_z) > t_z \qquad \forall z \in E_\varepsilon^{\mathsf{c}} , \\ Z(z, t_z) \in \partial E , \\ Z(z,t) \in E \qquad \forall (z,t) \in \partial E \times \mathbb{R}_{>0} . \end{gathered}$$

The continuity of the functions $(z,t) \in \mathbb{R}^m \times \mathbb{R} \mapsto Z(z,t) \in \mathbb{R}$ and $z \in E_\varepsilon^{\mathsf{c}} \mapsto t_z \in [-\varepsilon, +\infty[$ implies that this function $\phi$ is continuous at least on $\mathbb{R}^m \setminus \partial E_\varepsilon$. Also, for any $z$ in $\partial E_\varepsilon$, $t_z$ is defined and equal to $-\varepsilon$ (see (3)). So, for any strictly positive real number $\eta$, there exists a real number $r$ such that :

$$\begin{array}{rcll} |t_y + \varepsilon| & \leq & \eta & \forall y \in \mathcal{B}_r(z) , \\ \nu(t_y) + \varepsilon & \leq & \eta & \forall y \in \mathcal{B}_r(z) , \\ \phi(y) & = & y & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon , \\ \phi(y) & = & Z(y, t_y + \nu(t_y)) & \forall y \in \mathcal{B}_r(z) \cap E_\varepsilon^{\mathsf{c}} . \end{array}$$

Since we have :

$$\phi(z) = Z\left(z, t_z + \nu(t_z)\right) = Z\left(z, -\varepsilon + \nu(-\varepsilon)\right) = z ,$$

we conclude that $\phi$ is also continuous at $z$.

By differentiating, we obtain :

– at any interior point $z$ of $(E_\varepsilon)^{\mathsf{c}}$

$$\frac{\partial\phi}{\partial z}(z) = \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z)))\frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) ;$$

– at any $z$ in $E_\varepsilon$ (which is open) $\frac{\partial\phi}{\partial z}(z) = I$. Also, for any $z$ in $\partial E_\varepsilon$, we have :

$$\begin{aligned} \frac{\partial Z}{\partial z}(z, t_z + \nu(t_z)) + \chi(Z(z, t_z + \nu(t_z)))\frac{\partial t_z}{\partial z}(z)(1 + \nu'(t_z)) &= \frac{\partial Z}{\partial z}(z, 0) + \chi(Z(z,0))\frac{\partial t_z}{\partial z}(z)(1 - 1) , \\ &= I . \end{aligned}$$

This implies that $\phi$ is $C^1$ on $\mathbb{R}^m$.

We now show that $\phi$ is invertible. Let $y$ be arbitrary in $E \cap E_\varepsilon^{\mathsf{c}} = E \cap \Sigma$. There exists $t_y$ in $[-\varepsilon, 0[$. The function $\nu$ being strictly monotonic, $\nu^{-1}(t_y)$ exists and is in $[-\varepsilon, +\infty[$. This allows us to define properly $\phi^{-1}$ as :

$$\phi^{-1}(y) = \begin{cases} Z\left(y, t_y - \nu^{-1}(-t_y)\right), & \text{if } y \in E \backslash E_\varepsilon \\ y, & \text{if } y \in E_\varepsilon \end{cases} \tag{5}$$

This function is an inverse of $\phi$ as can be seen be reverting the flow induced by $\chi$ when needed. Also, with the same arguments as before, we can prove that it is $C^1$.

This implies that $\phi$ is a diffeomorphism from $\mathbb{R}^m$ to $E$.

Besides, the functions $z \mapsto Z(z,t)$ for $t > 0$, $z \mapsto t_z$ and $\nu$ being $C^s$, $\phi$ is $C^s$ at any interior point of $(E_\varepsilon)^{\mathsf{c}}$. By continuity of $\nu^{(r)}$ for $r \leq s$, it can be verified that $\phi$ is also $C^s$ on the boundary $\partial E_\varepsilon$. So, $\phi$ is a $C^s$-diffeomorphism from $\mathbb{R}^m$ to $E$.

Finally, we note that, for any point $z_\varepsilon$ in $\partial E_\varepsilon$, there exists a point $z$ in $\partial E$ satisfying :

$$|z_\varepsilon - z| \;=\; \left| \int_0^\varepsilon \chi(Z(z,s))ds \right| \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \;.$$

And conversely, for any $z$ in $\partial E$, there exist $z_\varepsilon$ in $\partial E_\varepsilon$ satisfying :

$$|z_\varepsilon - z| \;=\; \left| \int_0^\varepsilon \chi(Z(z,s))ds \right| \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \;.$$

It follows that

$$d_H(\partial E_\varepsilon, \partial E) \;\leq\; \varepsilon \sup_\zeta |\chi(\zeta)| \tag{6}$$

and $\varepsilon$ may be chosen as small as needed. □

Two direct consequences of Lemma 1 are :

**Corollary 1** *Let $\psi : E \to \psi(E)$ be a diffeomorphism, with $E$ satisfying $(\mathcal{C})$. For any $\varepsilon$ strictly positive, there exist an open subset $E_\varepsilon$ of $E$ and a diffeomorphism $\psi_e$: $\mathbb{R}^m \to \psi(E)$ satisfying :*

$$\begin{aligned} d_H(\partial E_\varepsilon, \partial E) \;&\leq\; \varepsilon \;, \\ \psi_e(z) = \psi(z) \qquad &\forall z \in E_\varepsilon \;. \end{aligned}$$

**Proof :** With $\phi$ given by Lemma 1, we pick $\psi_e \;=\; \psi \circ \phi$. □

**Corollary 2** *Let $\psi : E \to \psi(E)$ be a diffeomorphism, with $\psi(E)$ satisfying $(\mathcal{C})$. For any $\varepsilon$ strictly positive, there exist an open subset $\psi(E)_\varepsilon$ of $\psi(E)$ and a diffeomorphism $\psi_e$: $E \to \mathbb{R}^n$ satisfying :*

$$\begin{aligned} d_H(\partial\left(\psi(E)_\varepsilon\right), \partial\psi(E)) \;&\leq\; \varepsilon \;, \\ \psi_e(z) = \psi(z) \qquad &\forall z \in \psi^{-1}\left(\psi(E)_\varepsilon\right) \;. \end{aligned}$$

**Proof :** With $\phi$ given by Lemma 1 from $\mathbb{R}^n$ to $\psi(E)$, we pick $\psi_e = \phi^{-1} \circ \psi$. □

**Remark 2** In Corollary 2, $\psi$ being a diffeomorphism on an open set $E$, we know that the image of any compact subset $K$ of $E$ is a compact subset of $\psi(E)$ which is also open by Brouwer's invariance theorem. Therefore, with (6), we can find $\varepsilon$ such that $\psi(K) \subset (\psi(E))_\varepsilon$ and thus, $\psi_e(z) = \psi(z)$ for all $x$ in $K$.

## 1.2 Diffeomorphism extension from a ball

Let $R > 0$. We denote $\mathcal{B}_R$ the open ball in $\mathbb{R}^n$ of radius $R$ and centered at 0.

**Lemma 2** *Consider a $C^2$ diffeomorphism $\psi : \mathcal{B}_R \to \psi(\mathcal{B}_R) \subset \mathbb{R}^n$. For any $\varepsilon$ strictly positive, there exists a diffeomorphism $\psi_e : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi_e(x) = \psi(x)$ for all $x$ in $\mathtt{cl}(\mathcal{B}_{R-\varepsilon})$.*

**Proof :** Without loss of generality we may assume that $\psi(0) = 0$. Consider the mapping $\varphi : \mathcal{B}_R \times [0, 1] \to \mathbb{R}^n$ defined as

$$\varphi(x,t) = \left(\frac{\partial \psi}{\partial x}(0)\right)^{-1} \frac{\psi(xt)}{t} \ , \ \varphi(x,0) = x \ .$$

Note that for all $t$ the mapping $\varphi_t(x) = \varphi(x,t)$ is a diffeomorphism from $\mathcal{B}_R$ toward $\varphi_t(\mathcal{B}_R)$. Indeed, given $x_a$ and $x_b$ such that $\varphi(x_a, t) = \varphi(x_b, t)$ it yields $\psi(x_a t) = \psi(x_b t)$. Note that the couple $(x_a t, x_b t)$ is in $\mathcal{B}_R$. The mapping $\psi$ being injective on this set, it yields $x_a = x_b$. Moreover,

$$\frac{\partial \varphi_t}{\partial x}(x) = \left(\frac{\partial \psi}{\partial x}(0)\right)^{-1} \frac{\partial \psi}{\partial x}(xt) \ , \ t > 0 \ , \ \frac{\partial \varphi_0}{\partial x}(x) = \mathrm{Id}$$

Hence, this mapping is full rank in $\mathcal{B}_R$. Consequently, for all $t$ in $[0, 1]$, this mapping is a diffeomorphism $\mathcal{B}_R$ toward $\varphi_t(\mathcal{B}_R)$. Consequently, for all $t$ in $[0, 1]$ we can introduce $\varphi_t^{-1}$ its inverse map.

Note moreover that

$$\dot{\overbrace{\varphi(x,t)}} = \frac{\partial \varphi}{\partial t}(x,t) = \left(\frac{\partial \psi}{\partial x}(0)\right)^{-1} \rho(x,t)$$

where $\rho$ is the function given as

$$\rho(x,t) = \frac{1}{t^2}\left[\frac{\partial \psi}{\partial x}(xt)xt - \psi(xt)\right] \ , \ \rho(x,0) = \frac{1}{2}x'\left(\frac{\partial^2 \psi}{\partial x \partial x}(0)\right)x$$

The mapping $\psi$ being $C^2$, and $\psi(0) = 0$ it yields,

$$\psi(xt) = \frac{\partial \psi}{\partial x}(0)xt + x'\left(\frac{\partial^2 \psi}{\partial x \partial x}(0)\right)x\frac{t^2}{2} + \circ(t^2)$$

and,

$$\lim_{t \to 0} \frac{\frac{\partial \psi}{\partial x}(xt)x - \frac{\partial \psi}{\partial x}(0)x}{t} = x'\left(\frac{\partial^2 \psi}{\partial x \partial x}(0)\right)x$$

Hence, the function $\rho$ is well defined and locally Lipschitz. Consequently, $\varphi(x,t)$ is (the unique) solution of the time varying system defined for $(z,t)$ in $\varphi_t(\mathcal{B}_R)\times[0,1]$ as

$$\dot{z}=\left(\frac{\partial\psi}{\partial x}(0)\right)^{-1}\rho\left(\varphi_t^{-1}(z),t\right)$$

This time varying system can be extended to $\mathbb{R}^n$ as

$$\dot{z}=\left\{\begin{array}{l}0\ ,\ z\notin\varphi_t\left(\mathcal{B}_R\right)\\ \chi\left(\varphi_t^{-1}(z)\right)\left(\frac{\partial\psi}{\partial x}(0)\right)^{-1}\rho\left(\varphi_t^{-1}(z),t\right)\ ,\ z\in\varphi_t\left(\mathcal{B}_R\right)\end{array}\right. \tag{7}$$

where $\chi:\mathbb{R}^n\to\mathbb{R}^+$ is a $C^1$ function such that

$$\chi(x)=\left\{\begin{array}{l}0\ ,\ x\notin\mathcal{B}_R\\ 1\ ,\ x\in\mathcal{B}_{R-\varepsilon}\end{array}\right.$$

Notice that the vector field is zero outside $S=\bigcup\limits_{\substack{z\in\mathcal{B}_R\\ t\in[0,1]}}\varphi_t(z)$, which is a compact set. Indeed, if it is not, one can construct $(x_n)$ and $(t_n)$, such that for all $n\in\mathbb{N}$,

$$\frac{|\psi(x_nt_n)|}{t_n}\geq n.$$

Since $\mathtt{cl}(\mathcal{B}_R)$ is compact, we extract $x_{\sigma(n)}t_{\sigma(n)}$ tending towards $x^*\in\mathtt{cl}(\mathcal{B}_R)$. Necessarily, $t_{\sigma(n)}$ tends to 0 and $x*=0$ since $(x_n)$ is bounded. But this is impossible because, $\frac{\psi(x_nt_n)}{t_n}$ is equivalent to $\frac{\partial\psi}{\partial x}(0)x_n$ around zero.
Therefore,the maximal solutions to this system are defined (at least) in $[0,1]$ for all initial conditions, and through backward integration, we obtain that $\varphi(\mathbb{R}^n,1)=\mathbb{R}^n$.

We finally consider

$$\psi_e(x)=\frac{\partial\psi}{\partial x}(0)Z(x,1)$$

where $Z(x,1)$ is the solution of the system (7) evaluated at time 1 and initiated from $x$ at time 0. This mapping being a linear transformation of a (time varying) flow, it is a diffeomorphism, and $\psi_e(\mathbb{R}^n)=\mathbb{R}^n$. Note moreover that for all $x\in\mathcal{B}_{R-\varepsilon}$, we have

$$\psi_e(x)=\frac{\partial\psi}{\partial x}(0)\varphi(x,1)=\psi(x)\ .$$

□

## 2 Proof of Theorem 1

Consider a diffeomorphism $\psi$: $\mathcal{D}\to\psi(\mathcal{D})\subsetneq\mathbb{R}^m$. We want to extend the image of $\psi$ to $\mathbb{R}^m$, i-e find a diffeomorphism $\psi_e:\mathcal{D}\to\mathbb{R}^m$ such that:

- $\psi_e(\mathcal{D})=\mathbb{R}^m$

- $\psi_e(z) = \psi(z)$ for all $z$ in $K \subset \mathcal{D}$, where $K$ is any compact subset of $\mathcal{D}$.

Let us successively study the following two cases :

- **First case:** $\psi(\mathcal{D})$ **satisfies** $(\mathcal{C})$ The result follows directly from Corollary 2 and Remark 2. In practice, the reader may find an explicit construction of such an extension in the proof of Lemma 1.

- **Second case:** $\mathcal{D}$ **is** $C^2$**-diffeomorphic to** $\mathbb{R}^m$ **and** $\psi$ **is** $C^2$**:**

  Let $\phi_1 : \mathcal{D} \to \mathbb{R}^m$ denote the corresponding diffeomorphism. Let $R_1$ be a strictly positive real number such that the open ball $\mathcal{B}_{R_1}(0)$ contains $\phi_1(K)$. Let $R_2$ be a real number strictly larger than $R_1$. With Lemma 1 again, and since $\mathcal{B}_{R_2}(0)$ verifies condition $(\mathcal{C})$, there exists of $C^2$-diffeomorphism $\phi_2 : \mathcal{B}_{R_2}(0) \to \mathbb{R}^m$ satisfying

  $$\phi_2(z) \;=\; z \qquad \forall z \in \mathcal{B}_{R_1}(0)\ .$$

  At this point, we have obtained a $C^2$-diffeomorphism $\phi = \phi_2^{-1} \circ \phi_1 : \mathcal{D} \to \mathcal{B}_{R_2}(0)$. Consider $\lambda = \psi \circ \phi^{-1} : \mathcal{B}_{R_2}(0) \to \psi(\mathcal{D})$. According to Lemma 2, we can extend $\lambda$ to $\lambda_e : \mathbb{R}^m \to \mathbb{R}^m$ such that $\lambda_e = \psi \circ \phi^{-1}$ on $\mathcal{B}_{R_1}(0)$. Finally, consider $\psi_e = \lambda_e \circ \phi_1 : \mathcal{D} \to \mathbb{R}^m$. Since, by construction of $\phi_2$, $\phi = \phi_1$ on $\phi_1^{-1}(\mathcal{B}_{R_1}(0))$ which contains $K$, we have $\psi_e = \psi$ on K.

**Remark 3** Note that the second construction is more complex than the first one, for it contains several extensions and in particular that of Lemma 2 which is difficult to implement. For this reason, one may prefer applying the first case whenever it is possible.

# References


[1] P. Bernard, L. Praly, V.Andrieu, Tools for observers based on dynamic extension, Conference on Decision and Control, Under review

[2] V. Andrieu, J.-B. Eytard, L. Praly, Dynamic extension without inversion for observers, Conference on Decision and Control, 2014

[3] S. Gonnord, N. Tosel. Calcul Diffrentiel. Ellipses,1998.

[4] W. Hahn : *Stability of Motion.* Springer-Verlag, 1967

[5] J. Milnor, Lectures on the $h$-cobordism Theorem. Notes by L. Siebenmann and J. Sondow. Princeton Universiy Press, 1965.